On Construction of Modified Class of Estimators for Population Variance using Auxiliary Attribute

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Abstract: In this paper, an improved estimator for population variance has been proposed to improve the log-type estimators proposed by Kumari et al. (2019). The properties of proposed estimators are derived up to the first order of approximation. The proposed estimator found to be better than the existing estimators in the sense of mean squared error and percent relative efficiency. A numerical study is included to support the use of the suggested classes of estimators.

Keywords: Population Variance Estimators in The Sense of Mean Squared Error

I. INTRODUCTION

In sample surveys, auxiliary information always plays a vital role in better estimation of the parameter under investigation. This information can be used to improve the precision of the estimators. The suitable utilization of this auxiliary information can reduce the MSE of the sample mean, thus resulting in more efficient estimators. This paper, is an attempt to extend the powerful Searls approach to the traditional estimators using auxiliary information regarding to variables in simple random sampling. Many authors like, Singh et al. (1973), Das and Tripathi (1978), Sisodia and Dwivedi (1981), Isaki (1983), Bahl and Tuteja (1991), Prasad and Singh (1992), Swain (1994), Garcia and Cebrian (1996), Upadhaya and Singh (2001), Kalidar and Cingi (2006a, 2006b); Gupta and Shabbir (2006, 2007), Yadav and Kadilar (2013, 2014) had proposed an improved ratio estimators using Searls type estimators. Recently, Bhushan et al. (2017) among others; Kumari et al. (2019) have made the use of logarithmic relationship between the auxiliary attribute and study variable as logarithmic function which is very common in various branches of science as well as non-science disciplines. In this paper, some improved logarithmic estimators are proposed for improving the efficiency of the Kumari (2019) estimators as these classes of estimators are expected to improve the mean squared error. The proposed estimators would work considerably well in case when the study variable is logarithmically related to the auxiliary attribute.

Consider a finite population \( U = U_1, U_2, \ldots, U_N \) of size \( N \) from which a sample of size \( n \) is drawn according to simple random sampling without replacement (SRSWOR). Let \( y_i \) and \( f_i \) denotes the values of the study and auxiliary attribute for the \( ith \) unit \( (i = 1,2,\ldots,N) \) of the population. Further, let \( \bar{y} \) and \( \bar{f} \) be the sample means and

\[
\begin{align*}
\bar{y} &= \frac{1}{n} \sum_{i=1}^{n} y_i \\
\bar{f} &= \frac{1}{n} \sum_{i=1}^{n} f_i \\
s_y^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \\
s_f^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (f_i - \bar{f})^2 \\
\sigma^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \\
\sigma_f^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (f_i - \bar{f})^2
\end{align*}
\]

be the sample variance of the study and auxiliary attribute respectively up to the first order of approximation, are given as

\[
\begin{align*}
t_0 &= s_y^2 \\
t_1 &= s_y^2 \left( \frac{S_f^2}{S_f^2} \right) \\
t_2 &= s_y^2 \left( \frac{S_f^2}{S_f^2} \right)
\end{align*}
\]

2.1. Conventional Variance Estimator

The bias and variance of \( t_0 \) to the first order of approximation, are given as

\[
\begin{align*}
B(t_0) &= 0 \\
V(t_0) &= S_y^4 I
\end{align*}
\]

2.2. Isaki Ratio Estimator

The bias and MSE of \( t_1 \) to the first degree of approximation are given as

\[
\begin{align*}
B(t_1) &= S_y^2 I [b_{2y}^* - I_{22yf}] \\
V(t_1) &= S_y^4 I \left[ b_{2y}^* + b_{2f}^* - 2I_{22yf} \right]
\end{align*}
\]

2.3. Conventional Product Estimator

The bias and MSE of \( t_2 \) up to the first order of approximation are given as

\[
\begin{align*}
B(t_2) &= S_y^4 I I_{22}^* \\
V(t_2) &= S_y^4 I \left[ b_{2y}^* + b_{2f}^* + 2I_{22yf} \right]
\end{align*}
\]

2.4. Isaki Regression Estimator

Isaki (1983) suggested the following regression estimator for population variance

\[
t_3 = s_y^2 + b(S_f^2 - s_f^2)
\]

where \( b \) is a sample regression coefficient whose population regression coefficient is \( \beta \).

II. ESTIMATORS AVAILABLE IN LITERATURE

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The bias and MSE of $t_3$ to the first degree of approximation are given as

\[ B(t_3) = 0 \]
\[ MSE(t_3) = S_y^4 I \left[ \frac{n b_{2y}^*}{n + b_{2y}^*} - \frac{I_{22yf}^*}{b_{2f}^*} \right] \]

2.5. Singh et al. Estimator
Singh et al. (1973) Considered the Following Estimator

\[ t_4 = \alpha_4 \frac{s_y^2}{s_f^2} \]

Where $\alpha_4$ is a Searl (1964) constant. The optimum value of Searls constant is $\alpha_4 = n / (n + b_{2y}^*)$ for which the mean squared error is minimum.

\[ MSE(t_4)_{opt} = S_y^4 I \left[ \frac{n b_{2y}^*}{n + b_{2y}^*} \right] \]

2.6. Das and Tripathi Estimator

\[ t_5 = s_y^2 \left[ \frac{S_f^2}{S_f^2 + \alpha_5 (S_y^2 - S_f^2)} \right] \]

Where $\alpha_5$ is a constant. The bias and MSE of $t_5$ to the first degree of approximation is given as

\[ B(t_5) = S_y^2 I \left[ \frac{\alpha_5 b_{2y}^* - \alpha_5 I_{22yf}^*}{s_f^2 + \alpha_5 (s_y^2 - s_f^2)} \right] \]
\[ V(t_5) = S_y^4 I \left[ \frac{b_{2y}^* + \alpha_5 b_{2f}^* - 2 \alpha_5 I_{22yf}^*}{s_f^2 + \alpha_5 (s_y^2 - s_f^2)} \right] \]

The MSE of $t_5$ is optimum for $\alpha_5 = I_{22}^* / b_{2f}^*$ and is given by

\[ MSE(t_5)_{opt} = S_y^4 I \left[ \frac{b_{2y}^* - I_{22yf}^*}{b_{2f}^*} \right] \]

2.7. Prasad and Singh Estimator
Prasad and Singh (1992) introduced the following estimator

\[ t_6 = \alpha_6 \left( \frac{s_y^2 S_f^2}{s_f^2} \right) \]

where $\alpha_6$ is a Searls content. The bias and MSE of $t_6$ to the first degree of approximation is given as

\[ B(t_6) = S_y^2 I \left[ \alpha_6 (n + b_{2f}^* - I_{22yf}^*) - n \right] \]
\[ V(t_6) = S_f^4 I \left[ b_{2y}^* + \alpha_6 S_y^2 b_{2f}^* - 2 \alpha_6 S_y^2 I_{22yf}^* \right] \]

The MSE of $t_6$ is optimum for $\alpha_6 = (n + b_{2f}^* - I_{22yf}^*) / (n + b_{2y}^* + 3b_{2f}^* - 4I_{22yf}^*)$ and is given as

\[ MSE(t_6)_{opt} = S_y^4 I \left[ n - \frac{(n + b_{2f}^* - I_{22yf}^*)^2}{n + b_{2y}^* + 3b_{2f}^* - 4I_{22yf}^*} \right] \]

2.8. Garcia and Cebrian estimator
Garcia and Cebrian (1996) introduced the following estimator

\[ t_7 = \frac{s_y^2 (S_f^2)}{S_f^2} \]

where $\alpha_7$ is a Searls content.

The bias and MSE of $t_7$ to the first degree of approximation is given as

\[ B(t_7) = S_f^2 I \left[ \alpha_7 (n + b_{2y}^* - I_{22yf}^*) \right] \]
\[ V(t_7) = S_y^4 I \left[ b_{2y}^* + \alpha_7 b_{2f}^* - 2 \alpha_7 I_{22yf}^* \right] \]

The MSE of $t_7$ is optimum for $\alpha_7 = I_{22}^* / b_{2f}^*$ and is given as

\[ MSE(t_7)_{opt} = S_y^4 I \left[ b_{2y}^* - \frac{I_{22yf}^*}{b_{2f}^*} \right] \]

2.9. Upadhaya and Singh Estimator
Upadhaya and Singh (2001) suggested following estimator

\[ t_8 = s_y^2 + \alpha_6 (S_f^2 - S_y^2) \]

where $\alpha_6$ is a constant. The MSE of $t_8$ to the first degree of approximation is given as

\[ V(t_8) = S_y^4 I \left[ b_{2y}^* + \alpha_6 S_y^2 b_{2f}^* - 2 \alpha_6 S_y^2 I_{22yf}^* \right] \]

The MSE of $t_8$ is optimum for $\alpha_8 = S_y^2 I_{22yf}^* / S_f^2 b_{2f}^*$ and is given as

\[ MSE(t_8)_{opt} = S_y^4 I \left[ b_{2y}^* - \frac{I_{22yf}^*}{b_{2f}^*} \right] \]

2.10. Shabbir and Gupta (2006) estimator
Shabbir and Gupta (2006) proposed the Following Estimator

\[ t_9 = \lambda t_m \]

where $\lambda$ is a Searls (1964) constant whose value is to be determined later. Here $t_m$ is a combination of Singh et al. (1973), Prasad and Singh (1992) and is defined as
where \( K_1 \) and \( K_2 \) are the weights such that \( K_1 + K_2 = 1 \). The bias and MSE of \( t_9 \) to the first degree of approximation is given as

\[
B(t_9) = S_y^2 \left[ \lambda \left\{ n + K_2 (b_{22}^* - I_{22}^*) \right\} - n \right]
\]

\[
V(t_9) = S_y^4 \left[ \lambda^2 \left(n + b_{2y}^* + K_2 b_{22}^* + 2 K_2 b_{22}^* - 4 K_2 I_{22}^* \right) - 2 \lambda \left(n + K_2 b_{2f}^* - K_2 I_{22}^* \right) + n \right]
\]

The MSE of \( t_9 \) is optimum for

\[
\lambda = \frac{n + K_2 (b_{2f}^* - I_{22}^*)}{n + b_{2y}^* + K_2 b_{22}^* + 2 K_2 b_{22}^* - 4 K_2 I_{22}^*}
\]

and

\[
K_2 = \frac{I_{22}^*}{b_{2f}^*}
\]

\[
MSE(t_9)_{opt} = S_y^4 I \left[ n - \left\{ \frac{n + I_{22}^* - \frac{I_{22}^*}{b_{2f}^*}}{b_{2f}^*} \right\}^2 \right]
\]

2.11. 
Shabbir and Gupta (2007) estimator

\[
t_{10} = k_1 S_y^2 + k_2 \left(S_f^2 - S_y^2\right) e f p \left(\frac{S_f^2 - S_y^2}{S_f^2 + S_y^2}\right)
\]

where \( k_1 \) and \( k_2 \) are suitably chosen constants.

**Situation 1.** \( k_1 + k_2 = 1 \)

The bias and MSE of \( t_{10} \) the first degree of approximation are given as

\[
B(t_{10}) = (k_1 - 1) \left(S_y^2 - \frac{1}{2} S_f^2 I_{22}^* \right) + k_1 S_y^2 \left[3 \frac{S_f^*}{S_y} - \frac{1}{2} I_{22}^* \right]
\]

\[
V(t_{10}) = S_y^4 I \left[ k_1^2 \left\{ n + b_{2y}^* + S_f^2 b_{2f}^* + \frac{1}{4} b_{2f}^* - I_{22}^* + 2 S_y^2 \left(I_{22}^* - \frac{1}{2} b_{2f}^* \right) \right\} \right]
\]

\[
- 2 S_y^4 I k_1 \left\{ n + S_f^2 + S_y^2 \left(2 I_{22}^* - \frac{1}{2} b_{2f}^* \right) b_{2f}^* - K_2 I_{22}^* \right\} + n + S_y^2 S_f^2 \right]
\]

The optimum value of \( k_1 \) which minimizes MSE \( (t_{10}) \) is given as

\[
k_1 = \frac{A_1 + A_3}{A_1 + A_2 + 2 A_3}, \text{ where } A_1 = n + \frac{S_y^4}{S_f^2} b_{2f}^* \text{, } A_2 = b_{2y}^* + \frac{1}{4} b_{2f}^* - I_{22}^* \text{, } A_3 = S_f^2 \left(I_{22}^* - \frac{1}{2} b_{2f}^* \right)
\]
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\[ MSE(t^*_{10})_{opt} = S^4_y I \left[ A_1 - \frac{(A_1 + A_3)^2}{A_1 + A_2 + 2A_3} \right] \]

**Situation 2.** Unconstrained choice of \( k_1 \) and \( k_2 \)

The bias and MSE of \( t^*_{10} \) the first degree of approximation are given as

\[ B(t^*_{10}) = (k_1 - 1)S^2_y + \frac{1}{2} K_2 S^2_f I b^*_f + k_1 S^2_y I \left[ \frac{3}{8} b^*_f - \frac{1}{2} I^*_f \right] \]

\[ MSE(t^*_{10})_{opt} = S^4_y I \left[ \frac{Var(\hat{S}_{reg})}{1 + \frac{Var(\hat{S}_{reg})}{S^4_y}} \right] \]

2.12. **Kalidar and Cingi estimator**

Kalidar and Cingi (2006) suggested the following ratio type estimator

\[ t_{11} = \omega_1 S^2_y + \omega_2 \left( \frac{S^2_y}{S^2_f} \right)^T \]

where \( \omega_1 + \omega_2 = 1 \)

The optimum bias and MSE of \( t_{11} \) are given as

\[ B(t_{11}) = S^2_y \left[ (\omega_1^2 - 1) + \omega_1^2 \lambda I b^*_f \right] \]

\[ MSE(t_{11}) = S^4_y I \left[ \lambda^2 (n + b^*_y + K_2 b^*_f + 2K_2 b^*_f - 4K_2 I^*_f) - 2\lambda (n + K_2 b^*_f - K_2 I^*_f) + n \right] \]

The MSE of \( t_{11} \) is optimum for \( \lambda = \frac{n + K_2 (b^*_y - I^*_f)}{n + b^*_y + K_2 b^*_f + 2K_2 b^*_f - 4K_2 I^*_f} \) and

\[ K_2 = \frac{I^*_f}{b^*_f} \]

is given by

\[ MSE(t_{11})_{opt} = S^4_y I \left[ n - \frac{n + I^*_f - I^*_f}{b^*_f} \right]^2 \left( \frac{n + b^*_y + 2I^*_f - 3I^*_f}{b^*_f} \right) \]


Yadav and Kadilar (2013) introduced the following estimator

\[ t_{12} = S^2_y e_f p \left[ 1 - \frac{a_{12} S^2_f}{S^2_f + (a_{12} - 1) S^2_f} \right] \]

where \( a_{12} \) is a constant. The optimum MSE of \( t_{12} \) to the first degree of approximation is given as

Yadav and Kadilar (2014) introduced the following ratio-product-ratio estimator

\[ t_{a,b} = \alpha \left[ \frac{(1 - \beta) s_f^2 + \beta S_f^2}{\beta s_f^2 + (1 - \beta) S_f^2} \right] s_y^2 + (1 - \alpha) \left[ \frac{\beta S_f^2 + (1 - \beta) S_f^2}{(1 - \beta) s_f^2 + \beta S_f^2} \right] s_y^2 \]

where \( \alpha \) and \( \beta \) are real constant. The optimum MSE of \( t_{a,b} \) to the first degree of approximation is given as

\[ MSE(t_{12})_\text{opt} = S_y^4 I \left\{ \frac{b_{2y}^* - I_{2y}^*}{b_{2f}^*} \right\} \]

### III. THE PROPOSED ESTIMATORS

We propose the following new classes of log-type estimators for the population variance \( S_y^2 \) as:

\[ T_1 = \frac{w_1 S_y^2 \left[ 1 + \log \left( \frac{S_f^2}{S_y^2} \right) \right]}{\beta s_f^2 + (1 - \beta) S_f^2} \]  
\[ T_2 = \frac{w_2 S_y^2 \left[ 1 + a_2 \log \left( \frac{S_f^2}{S_y^2} \right) \right]}{(1 - \beta) s_f^2 + \beta S_f^2} \]  
\[ T_3 = \frac{w_3 S_y^2 \left[ 1 + a_3 \log \left( \frac{S_f^2}{S_y^2} \right) \right]}{(1 - \beta) s_f^2 + \beta S_f^2} \]  
\[ T_4 = \frac{w_4 S_y^2 \left[ 1 + a_4 \log \left( \frac{S_f^2}{S_y^2} \right) \right]}{(1 - \beta) s_f^2 + \beta S_f^2} \]

where

\[ s_f^2 = a s_f^2 + b \]
\[ S_f^2 = a S_f^2 + b \]

such that \( a \neq 0 \), \( b \) are either real numbers or functions of the known parameters of the auxiliary attribute \( f \) such as the standard deviations \( S_f \), coefficient of variation \( C_f \), coefficient of kurtosis \( b_f \), coefficient of skewness \( b_f \), and correlation coefficient \( \rho \) of the population. If \( a = 0 \), then the proposed estimator becomes the usual per unit variance estimator \( S_y^2 \). If \( a = 1 \), then the proposed estimator works as ratio type estimator and if \( a = -1 \), then the proposed estimator works as product type estimator having efficiency conditions equivalent to that of generalized product and ratio estimators respectively.

### IV. BIAS AND MSE OF PROPOSED ESTIMATORS

In this paper, the biases and the mean square error(s) of all the estimators are considered up to the terms of order \( n^{-1} \).

**Theorem 1.** The bias and MSE of \( T_1^* \) are given by

\[ Bias(T_1^*) = S_y^2 \left\{ w_1 \left[ 1 - a_1 I_{2y}^* + a_1 I_{2y}^* + \frac{a_1^2}{2} I_{2y}^* \right] - 1 \right\} \]
\[ MSE(T_1^*) = S_y^4 w_1^2 \left\{ 1 + \frac{1}{2} \left[ a_1 b_{2y}^* + a_1 b_{2y}^* - 4 a_1 b_{2y}^* + 2 a_1 b_{2y}^* \right] \right\} \]

\[ - 2 S_y^4 w_1 \left[ 1 + \frac{1}{2} \left[ a_1 b_{2y}^* + a_1 b_{2y}^* - 4 a_1 b_{2y}^* + 2 a_1 b_{2y}^* \right] \right] + 1 \]

Proof. Consider the estimator,
Squaring on both the sides and then taking expectation over the proposed estimator, we have

\[
S_y^4(w_1 - 1)^2 + w_1 S_y^2 \left[ E(\epsilon_0)^2 + a_1^2 E(\epsilon_1)^2 + 2a_1 E(\epsilon_0\epsilon_1) \right]
+ 2 S_y^4 w_1 (w_1 - 1) \left[ a_1 E(\epsilon_1)^2 - a_1 E(\epsilon_0\epsilon_1) + \frac{a_1^2}{2} E(\epsilon_1^2) \right]
\]

Using the results from Sukhatme and Sukhatme, we have

\[
E(\epsilon_0) = 0 = E(\epsilon_1), \quad E(\epsilon_0)^2 = I_1 b_{2y}, \quad E(\epsilon_1)^2 = I_2 b_{2f}, \quad E(\epsilon_0\epsilon_1) = I_1 b_{22y}, \quad E(\epsilon_0) = I_1 b_{22f}, \quad I_{22y} = I_{22f} = b_{2y} - 1, \quad I_{22y} = I_{22f} = 1, \quad I_{pq} = m_{pq}/m_{20}^2, \quad m_{pq} = \sum_{i=1}^{N} (Y_i - \bar{Y})^p (Y_i - \bar{Y})^q / N, \quad I = 1/N, \quad b_{2y} = m_{40}/m_{20}^2, \quad b_{2f} = m_{04}/m_{02}^2\]

are the coefficient of kurtosis of \(y\) and \(f\) respectively.

Now, substituting the above results in (4:1), we get

\[
MSE(T_1^*) = S_y^4 w_1 \left[ 1 + I(b_{2y}^* + 2a_1^2 b_{2f}^* - 4a_1 I_{22yf}^* + 2a_1 b_{2f}^*) \right]
- 2 S_y^4 w_1 \left[ 1 + I(a_1 b_{2f}^* + \frac{a_1^2 b_{2f}^*}{2} - a_1 I_{22yf}^*) \right] + 1
\]

Corollary 2. The optimum value of \(a_1\) and \(w_1\) are

\[
a_1 = \frac{I_{22y}^*}{b_{2f}}
\]

\[
w_1 = \frac{B_1}{A_1}
\]

where

\[
A_1 = 1 + I(b_{2y}^* + 2a_1^2 b_{2f}^* - 4a_1 I_{22yf}^* + 2a_1 b_{2f}^*)
\]

\[
B_1 = 1 + I(a_1 b_{2f}^* + \frac{a_1^2 b_{2f}^*}{2} - a_1 I_{22yf}^*)
\]

respectively. Also, the minimum mean squared error of \(T_1^*\) is

\[
MSE(T_1^*)_{opt} = S_y^4 I \left[ 1 - \frac{B_1^2}{A_1} \right]
\]

Proof: Obvious, using Theorem 1.

Theorem 3. The bias and MSE of \(T_2^*\) are given by
\[ \text{Bias} \left( T_2^* \right) = S_y^2 \left[ w_2 \left\{ 1 - a_2 \eta I_{22yf}^* + \frac{a_2^2}{2} I_{2f}^* \right\} - 1 \right] \]

\[ \text{MSE} \left( T_2^* \right) = S_y^4 \left[ w_2^2 \left[ 1 + I (b_{2y}^* + a_2^2 b_{2f}^* - 4a_2 I_{22yf}^* + a_2 b_{2f}^*) \right] - 2S_y^4 w_2 \left[ 1 + I \left( \frac{a_2 b_{2f}^*}{2} - a_2 I_{22yf}^* \right) \right] + 1 \right] \]

respectively.

Proof. Trivial and similar to derivation of Theorem 1, hence omitted.

**Corollary 4.** The optimum value of \( a_2 \) and \( w_2 \) are

\[ a_2 = \frac{I_{22yf}^*}{b_{2f}^*} \quad \text{and} \quad w_2 = \frac{B_2}{A_2} \]

where

\[ A_2 = 1 + I (b_{2y}^* + a_2^2 b_{2f}^* - 4a_2 I_{22yf}^* + a_2 b_{2f}^*) \]

\[ B_2 = 1 + I \left( \frac{a_2 b_{2f}^*}{2} - a_2 I_{22yf}^* \right) \]

Also, the minimum mean squared error of \( T_2^* \) is

\[ \text{MSE} \left( T_2^* \right)_{\text{opt}} = S_y^4 \left[ 1 - \frac{B_2^2}{A_2} \right] \]

**Theorem 5.** The bias and MSE of \( T_3^* \) are given by

\[ \text{Bias} \left( T_3^* \right) = S_y^2 \left[ w_1 \left\{ 1 - a_3 \eta I_{22yf}^* + a_3^2 \eta^2 I_{2f}^* + \frac{a_3^2}{2} \eta \eta^2 b_{2f}^* \right\} - 1 \right] \]

\[ \text{MSE} \left( T_3^* \right) = S_y^4 \left[ w_1^2 \left[ 1 + I (b_{2y}^* + a_3^2 \eta^2 b_{2f}^* - 4a_3 \eta b_{2f}^* + 2a_3 \eta^2 b_{2f}^*) \right] - 2S_y^4 w_1 \left[ 1 + I (a_3 \eta^2 b_{2f}^* + \frac{a_3^2}{2} \eta^2 b_{2f}^* - a_3 \eta I_{22yf}^*) \right] + 1 \right] \]

respectively.

Proof. Trivial and similar to derivation of Theorem 1, hence omitted.

**Corollary 6.** The optimum value of \( a_3 \) and \( w_3 \) is

\[ a_3 = \frac{I_{22yf}^*}{b_{2f}^*} \quad \text{and} \quad w_3 = \frac{B_3}{A_3} \]

where

\[ A_3 = 1 + I (b_{2y}^* + 2a_3^2 \eta^2 b_{2f}^* - 4a_3 \eta b_{2f}^* + 2a_3 \eta^2 b_{2f}^*) \]

\[ B_3 = 1 + I (a_3 \eta^2 b_{2f}^* + \frac{a_3^2}{2} \eta^2 b_{2f}^* - a_3 \eta I_{22yf}^*) \]

Also, the minimum mean squared error of \( T_3^* \) is
Theorem 7. The bias and MSE of $T_3^*$ are given by

$$\text{Bias} (T_3^*) = S_y^4 \left[ w_4 \left( 1 - a_4^\eta I_{22y}^* + \frac{a_4^2}{2} I_{22y}^* b_{2f}^* \right) - 1 \right]$$

$$\text{MSE} (T_3^*) = S_y^4 w_4^2 \left[ 1 + I (b_{2y}^* + a_4^2 \eta b_{2f}^* - 4a_4^\eta I_{22y}^* f + a_4^\eta b_{2f}^* ) - 2S_y^4 w_2 \left[ 1 + I \left( \frac{a_4^\eta b_{2f}^*}{2} - a_4^\eta I_{22y}^* f \right) \right] + 1 \right]$$

respectively.

Proof. Trivial and similar to derivation of Theorem 1, hence omitted.

Corollary 8. The optimum value of $a_4$ and $w_4$ is

$$a_4 = \frac{I_{22}^*}{b_{2f}^*} \quad \text{and} \quad w_4 = \frac{B_4}{A_4}$$

where

$$A_4 = 1 + I (b_{2y}^* + a_4^2 \eta b_{2f}^* - 4a_4^\eta I_{22y}^* f + a_4^\eta b_{2f}^* )$$

$$B_4 = 1 + I \left( \frac{a_4^\eta b_{2f}^*}{2} - a_4^\eta I_{22y}^* f \right) + 1$$

Also, the minimum mean squared error of $T_3^*$ is

$$\text{MSE} (T_3^*)_{\text{opt}} = S_y^4 \left[ 1 - \frac{B_4^2}{A_4} \right]$$

V. COMPARISON OF ESTIMATORS

In this section, we compare the proposed classes of estimators with some important estimators. The comparison will be in terms of their MSEs up to the order of $n^{-1}$. Let us define

$$C_1 = b_{2y}^* + b_{2f}^* - 2I_{22}^*, \quad C_2 = b_{2y}^* + b_{2f}^* - 2I_{22}^*, \quad D = b_{2y}^* b_{2f}^* - I_{22}^*, \quad E = \left[ \frac{nb_{2y}^*}{n + b_{2y}^*} \right], \quad F = \left[ \frac{(n + b_{2f}^* - I_{22}^*)^2}{n + b_{2y}^* + 3b_{2f}^* - 4I_{22}^*} \right],$$

$$G = \left[ n - \frac{\left( n + I_{22}^* - \frac{I_{22}^2}{b_{2f}^*} \right)^2}{n + b_{2y}^* + 2I_{22}^* - 3\frac{I_{22}^2}{b_{2f}^*}} \right], \quad H = \left[ A_1 - \frac{(A_1 + A_3)^2}{A_1 + A_2 + 2A_3} \right].$$
VI. EMPIRICAL STUDY

To compare the efficiency of the suggested class of estimator numerically, we considered nine natural data sets. The description of the population is given below.

Population 1. (Cochran (1977), Pg. no. 107)

\( y \) : number of persons per block

\( f \) : number of rooms per block

\( S_y^2 = 214.69, S_f^2 = 56.76, b_{2y}^* = 1.2387, b_{2f}^* = 1.3523, I_{2y} = 0.5432, C_f = 0.1450, \bar{f} = 58.8, \rho = 0.6515, n = 10. \)

Population 2. (Cochran (1977), Pg. no. 203)

\( y \) : actual weight of peaches on each tree

\( f \) : eye estimate of weight of peaches on each tree.

\( S_y^2 = 99.81, S_f^2 = 85.09, b_{2y}^* = 0.9249, b_{2f}^* = 1.2932, I_{2y} = 1.1149, C_f = 0.1621, \bar{f} = 56.9, \rho = 0.9937, n = 10. \)
On Construction of Modified Class of Estimators for Population Variance using Auxiliary Attribute

Population 3. (Sukhatme P. V. (1970), Pg. no. 185)
y : wheat acreage in 1937
f : wheat acreage in 1936
\[ S_y^2 = 26456.99, \quad S_f^2 = 22355.76, \quad b_{2y} = 2.1842, \quad b_{2f} = 1.2030, \quad I_{22} = 1.5597, \quad C_f = 0.5625, \quad \bar{f} = 265.8, \quad \rho = 0.977, \quad n = 10. \]

Population 4. (Singh D and Chaudhary F. S., Pg. no. 107). y : number of boats landing at a particular centre
f : catch of fish in quintals.
\[ S_y^2 = 201324.4, \quad S_f^2 = 396.8889, \quad b_{2y} = 0.9462, \quad b_{2f} = 0.6078, \quad I_{22} = 0.6333, \quad C_f = 0.7288, \quad \bar{X} = 27.3333, \quad \rho = 0.9308, \quad n = 4. \]

Population 5. (Singh D and Chaudhary F. S., Pg. no. 141).
y : number of bearing lime trees
f : area under lime (in acres)
\[ S_y^2 = 6564586.45, \quad S_f^2 = 1092.1024, \quad b_{2y} = 12.2574, \quad b_{2f} = 4.5788, \quad I_{22} = 6.7126, \quad C_f = 1.4273, \quad \bar{X} = 22.6209, \quad \rho = 0.9021, \quad n = 9. \]

Population 6. (Choudhary F. S. and Singh D., Pg. no. 176).
y : number of cows in milk enumerated
f : number of cows in milk in the previous year.
\[ S_y^2 = 332721.2079, \quad S_f^2 = 281472.7868, \quad b_{2y} = 6.2079, \quad b_{2f} = 5.0043, \quad I_{22} = 4.9528, \quad C_f = 0.8276, \quad \bar{X} = 641.05, \quad \rho = 0.8933, \quad n = 8. \]

Population 7. (Singh S., Pg. no. 324-325). y : approximate duration of sleep (in minutes)
f : age in years of the persons.
\[ S_y^2 = 3582.579, \quad S_f^2 = 85.2367, \quad b_{2y} = 1.6678, \quad b_{2f} = 1.2389, \quad I_{22} = 0.9961, \quad C_f = 0.1349, \quad \bar{X} = 67.2667, \quad \rho = -0.8552, \quad n = 9. \]

Population 8. (Singh S., Pg. no. 1114). y : approximate duration of sleep (in minutes)
f : age in years of the persons.
\[ S_y^2 = 0.0073, \quad S_f^2 = 0.0063, \quad b_{2y} = 2.6323, \quad b_{2f} = 2.4016, \quad I_{22} = 1.8351, \quad C_f = 1.2352, \quad \bar{X} = 0.1831, \quad \rho = 0.7789, \quad n = 11. \]

By using the above data set, the percent relative efficiency of the different estimator are given in Table 2.

In the above table, the relative efficiency of the proposed estimator is much better as compared to other estimators for all the data sets given here.

Table 1: PRE of the Estimators with Respect to \( l_0 \)

<table>
<thead>
<tr>
<th>Est.</th>
<th>Pop 1</th>
<th>Pop 2</th>
<th>Pop 3</th>
<th>Pop 4</th>
<th>Pop 5</th>
<th>Pop 6</th>
<th>Pop 7</th>
<th>Pop 8</th>
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<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
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<td>320.81</td>
<td>815.60</td>
<td>329.16</td>
<td>359.35</td>
<td>475.29</td>
<td>182.39</td>
<td>193.03</td>
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<td>( t_2 )</td>
<td>33.68</td>
<td>19.48</td>
<td>33.56</td>
<td>33.54</td>
<td>40.50</td>
<td>20.39</td>
<td>34.04</td>
<td>30.24</td>
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<tr>
<td>( t_3 )</td>
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<td>639.14</td>
<td>1347.98</td>
<td>330.39</td>
<td>507.23</td>
<td>475.29</td>
<td>192.40</td>
<td>214.00</td>
</tr>
<tr>
<td>( t_4 )</td>
<td>112.38</td>
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<td>121.42</td>
<td>30.91</td>
<td>236.19</td>
<td>177.59</td>
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<td>123.93</td>
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<tr>
<td>( t_5 )</td>
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<td>109.23</td>
<td>1347.98</td>
<td>330.96</td>
<td>507.23</td>
<td>475.29</td>
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<td>( t_6 )</td>
<td>112.95</td>
<td>639.14</td>
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<td>381.77</td>
<td>558.96</td>
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<td>( t_8 )</td>
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<td>349.88</td>
<td>528.70</td>
<td>559.08</td>
<td>220.35</td>
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</tr>
<tr>
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<td>1863.95</td>
<td>439.47</td>
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<td>2120.43</td>
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<td>311.67</td>
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VII. CONCLUSION

The present study extends the idea of Bhushan and Kumari (2019) regarding the effective use of auxiliary information if the relationship between the study variable and the auxiliary attribute is of logarithmic type. Further, the efficiency of the proposed estimators are compared with some conventional estimators and some recent estimators of Singh et al. (1973), Das and Tripathi (1978), Sisodia and Dwivedi (1981), Isaki (1983), Bahl and Tuteja (1991), Prasad and Singh (1992), Swain (1994), Garcia and Cebrian (1996), Upadhyaya and Singh (2001), Kalidar and Cingi (2006a, 2006b); Gupta and Shabbir (2006, 2007), Yadav and Kadilar (2013, 2014). The proposed estimator is most efficient than all the estimators. This study is also supported through an empirical study and the result of this study is quite encouraging.

REFERENCES


AUTHOR DETAILS

Dr. Chandni Kumari received the PhD degree in Statistics from Babasaheb Bhimrao Ambedkar University, Lucknow, India. Her research interests are in the areas of applied statistics. In particular, she has keen interest on Sample Surveys. She has published many research articles in reputed international and national journals.

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