

Vertex Polynomial of Middle, Line and Total Graphs of some standard Graphs

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Abstract: In this article, we have obtained vertex polynomial of Middle, Line and Total graphs of some standard graphs namely complete graph, star, path, cycle and wheel graph.

Keywords: Vertex polynomial, Middle graph, Line Graph, Total Graph

I. INTRODUCTION

In this paper, we have considered only finite graphs which are non-trivial, undirected, having no self-loops and having no parallel edges. Two edges of a graph are adjacent if they have a common vertex. If a vertex v is an end vertex of an edge e , then edge e is incident on vertex v . Vertices and edges are called elements of a graph and are neighbors if they are either incident or adjacent. $K_{1,n}$ represents a star. A wheel graph W_n and is defined as $W_n = K_1 + C_{n-1}$. A graph G is k -regular if each vertex has degree k . If $x = uv$ is an edge of a graph G , and w is not a vertex of G , then x is said to be subdivided if it is replaced by the edges uw and vw . For terminologies and notations, we refer [1]. The vertex polynomial of the graph G [2] is defined as $V(G; x) = \sum_{k=0}^{\Delta(G)} v_k x^k$, where $\Delta(G) = \max\{d(v) : v \in V(G)\}$ and v_k is the number of vertices of degree k . The roots of a vertex polynomial are called vertex polynomial roots.

II. MIDDLE GRAPH, LINE GRAPH, TOTAL GRAPH AND VERTEX POLYNOMIAL WITH AN EXAMPLE.

In this part, we list the definitions of Middle graph, Line graph and Total graph of a graph with an example. Also we show their vertex polynomials.

Let G be a graph with $|V| = m$ and $|E| = n$. Its line graph $L(G)$ contains vertices which are edges of G and edges in $L(G)$ are drawn between vertices if they have common vertex in G . Middle graph $M(G)$ has vertex set containing all vertices and all edges of G and edges of $M(G)$ between new vertices are drawn only when the edges corresponding in G are adjacent. Total graph $T(G)$ has vertex set containing all vertices of G and all edges of G and two vertices of total graph are adjacent if they happen to be neighbors in G . The following figures show a graph, its line graph, middle graph and total graph with their vertex polynomials.



Figure 1– Graph G .

In this graph, we find 1, 2 and 1 vertices of degrees 1, 2 and 3 respectively. Thus $V(G; x) = x + 2x^2 + x^3$. Now we consider its line graph as in Figure 2.

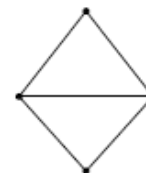


Figure 2– $L(G)$

We note that $L(G)$ has four vertices of which two are of degree 2 and remaining two are of degree 3. Thus $V(L(G); x) = 2x^2 + 2x^3$

Now, we consider middle graph $M(G)$ as shown in Figure 3.

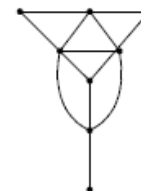


Figure 3– $M(G)$

The middle graph has one vertex of degree 1, two vertices of degree 2, one vertex of degree 3, two vertices of degree 4 and two vertices of degree 5. Thus $V(M(G); x) = x + 2x^2 + x^3 + 2x^4 + 2x^5$. Now we consider total graph $T(G)$ as shown in Figure 4.

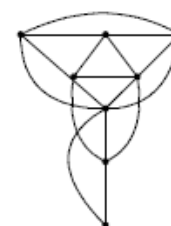


Figure 4– $T(G)$

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$T(G)$, the total graph has 1, 4, 2 and 1 vertices of degrees 2, 4, 5 and 6 respectively.

$$\text{Thus } V(T(G); x) = x^2 + 4x^4 + 2x^5 + x^6$$

III. MAIN RESULTS

Theorem 3.1: If $M(K_n)$ is middle graph of K_n , then

$$V(M(K_n); x) = nx^{n-1} + \frac{n(n-1)}{2}x^{2n-2}$$

Proof: We note that K_n has n vertices, whereas its middle graph $M(K_n)$ has $\frac{n^2+n}{2}$ vertices by definition of middle graph. We can partition vertex set of $M(K_n)$ into two disjoint subsets V_1 and V_2 as follows:

V_1 : Vertex set with $|V_1| = n$, degree being $(n-1)$ for each vertex and

V_2 : Vertex set with $|V_2| = \frac{n^2-n}{2}$, degree being $(2n-2)$ for each vertex.

$$\text{Hence } V(M(K_n); x) = nx^{n-1} + \frac{n(n-1)}{2}x^{2n-2}$$

Theorem 3.2: If $M(K_{1,n})$ is middle graph of $K_{1,n}$, then

$$V(M(K_{1,n}); x) = nx^{n+1} + x^n + nx \text{ for } n \geq 2$$

Proof: We note that $K_{1,n}$ has $(n+1)$ vertices, whereas its middle graph $M(K_{1,n})$ has $(2n+1)$ vertices by definition of middle graph. We can partition the vertex set of $M(K_{1,n})$ into disjoint subsets V_1, V_2 and V_3 as follows:

V_1 : Vertex set containing 1 vertex of degree n ,

V_2 : Vertex set with $|V_2| = n$, each vertex being pendant and

V_3 : Vertex set $|V_3| = n$, degree being $(n+1)$ for each vertex.

$$\text{Thus } V(M(K_{1,n}); x) = nx^{n+1} + x^n + nx$$

Theorem 3.3: If $M(P_n)$ is the middle graph of P_n , then

$$V(M(P_n); x) = 2x + (n-2)x^2 + (n-1)x^3, n \geq 3$$

Proof: We note, by definition of middle graph that, $M(P_n)$ contains $(2n-1)$ vertices. We can partition the vertex set of $M(P_n)$ into three disjoint subsets V_1, V_2 and V_3 as follows:

V_1 : Vertex set containing 2 vertices, each of degree 1,

V_2 : Vertex set with $|V_2| = n-2$, degree being 2 for each vertex and

V_3 : Vertex set with $|V_3| = n-1$, degree being 3 for each vertex.

$$\text{Thus } V(M(P_n); x) = 2x + (n-2)x^2 + (n-1)x^3$$

Theorem 3.4: If $M(C_n)$ is the middle graph of C_n , then

$$V(M(C_n); x) = nx^2 + nx^4$$

Proof: We note, by definition of middle graph that, $M(C_n)$ contains $2n$ vertices. We can partition the vertex set of $M(C_n)$ into V_1 and V_2 as follows:

V_1 : Vertex set containing n vertices of C_n , each of degree 2 and

V_2 : Vertex set containing n new vertices obtained by definition of middle graph, each of degree 4.

$$\text{Thus } V(M(C_n); x) = nx^2 + nx^4$$

Corollary 3.5: The vertex polynomial roots of $M(C_n)$ are 0, 0 and $\pm i$

Proof: In view of Theorem 3.4, vertex polynomial roots of $M(C_n)$ can be obtained by taking

$$nx^2 + nx^4 = 0 \Rightarrow x = 0, 0 \text{ and } \pm i$$

Theorem 3.6: If $M(W_n)$ is the middle graph of W_n , then

$$V(M(W_n); x) = (n-1)x^{n+2} + x^{n-1} + (n-1)x^6 + (n-1)x^3, \text{ for } n \geq 5$$

Proof: We note, by definition of middle graph that, $M(W_n)$ contains $(3n-2)$ vertices. We can partition vertex set of $M(W_n)$ into four disjoint subsets V_1, V_2, V_3 and V_4 as follows:

V_1 : Vertex set containing a single vertex of degree $(n-1)$,

V_2 : Vertex set with $|V_2| = n-1$, degree being 3 for each vertex,

V_3 : Vertex set $|V_3| = n-1$, degree being 6 for each vertex, and

V_4 : Vertex set $|V_4| = n-1$, degree being $n+2$ for each vertex.

$$\text{Thus } V(M(W_n); x) = (n-1)x^{n+2} + x^{n-1} + (n-1)x^6 + (n-1)x^3$$

Theorem 3.7: If $L(K_n)$ is the line graph of K_n , then

$$V(L(K_n); x) = \frac{n(n-1)}{2}x^n$$

Proof: Using definition of line graph, we easily understand that $L(K_n)$ is a $n-regular$ graph with $\frac{n^2-n}{2}$ vertices and every vertex will be of degree n .

$$\text{Thus } V(L(K_n); x) = \frac{n(n-1)}{2}x^n.$$

Theorem 3. 8: If $L(K_{1,n})$ is the line graph of $K_{1,n}$, then

$$V(L(K_{1,n}); x) = nx^{n-1}$$

Proof: By definition of line graph, of a graph, we easily understand that $L(K_{1,n})$ is a complete graph K_n and therefore every vertex will be of degree $(n-1)$.



Thus $V(L(K_{1,n}; x)) = nx^{n-1}$.

Theorem 3.9: If $L(P_n)$ is the line graph of P_n , then $V(L(P_n); x) = 2x + (n - 3)x^2, n \geq 3$

Proof: By definition of line graph, we easily understand that $L(P_n)$ is a path graph P_{n-1} .

Thus $V(L(P_n); x) = V(P_{n-1}; x)$.

It is easy to note that path graph P_{n-1} will have 2 pendant vertices and remaining $(n - 3)$ internal vertices will be of degree 2.

Thus $V(L(P_n); x) = 2x + (n - 3)x^2$.

Theorem 3.10: If $L(C_n)$ is the line graph of C_n , then is $V(L(C_n); x) = nx^2$.

Proof: By definition of line graph, we easily understand that $L(C_n) = C_n$ itself. As degree of every vertex of C_n is 2 and as it has n vertices, we find $V(L(C_n); x) = nx^2$.

Corollary 3.11: The vertex polynomial root of $L(C_n)$ is 0 with multiplicity 2.

Proof: In view of Theorem 3.10, vertex polynomial roots of $L(C_n)$ can be obtained by taking

$$nx^2 = 0 \Rightarrow x = 0, 0 \text{ as } n \neq 0$$

Theorem 3.12: If $L(W_n)$ is the line graph of W_n , then

$$V(L(W_n); x) = (n - 1)x^4 + (n - 1)x^n, \quad n \geq 4$$

Proof: We know, a wheel graph W_n will have n vertices and $(2n - 2)$ edges. The vertex set of $L(W_n)$ will have $(2n - 2)$ vertices and we can partition the vertex set into two disjoint subsets V_1 and V_2 as follows:

V_1 : Vertex set with $|V_1| = n - 1$, degree being 4 for each vertex

V_2 : Vertex set with $|V_2| = n - 1$, degree being n for each vertex

Thus $V(L(W_n); x) = (n - 1)x^4 + (n - 1)x^n$

Theorem 3.13: If $T(K_n)$ is the total graph of K_n , then

$$V(T(K_n); x) = \frac{n(n + 1)}{2} x^{2n-2}$$

Proof: We note that K_n will have n vertices and $\frac{n(n-1)}{2}$ edges. By definition of total graph of a graph, we note that, total graph $T(K_n)$ will have $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ vertices and each vertex will be of degree $(2n - 2)$.

Thus

$$V(T(K_n); x) = \frac{n(n + 1)}{2} x^{2n-2}$$

Theorem 3.14: If $T(K_{1,n})$ is the total graph of $K_{1,n}$, then

$$V(T(K_{1,n}); x) = nx^2 + nx^{n+1} + x^{2n}$$

Proof: We know a star graph $K_{1,n}$ will have $n + 1$ vertices and n edges. We note that total graph $T(K_{1,n})$ of star graph $K_{1,n}$ will have $(2n + 1)$ by the way in which total graph is defined and we can partition this the vertex set into 3 disjoint subsets V_1, V_2 and V_3 as follows:

V_1 : Vertex set containing a single vertex of degree $2n$,

V_2 : Vertex set with $|V_2| = n - 1$, degree being 2 for each vertex and

V_3 : Vertex set $|V_3| = n$, degree being 2 for each vertex.

Thus $V(T(K_{1,n}); x) = nx^2 + nx^{n+1} + x^{2n}$

Theorem 3.15: If $T(P_n)$ is the total graph of $T(P_n)$, then

$$V(T(P_n); x) = 2x^2 + 2x^3 + (2n - 5)x^4, n \geq 3$$

Proof: We know, a path graph P_n will have n vertices and $(n - 1)$ edges. By definition of total graph, we note $T(P_n)$ will have $(2n - 1)$ vertices and we can partition this vertex set into 3 disjoint subsets V_1, V_2 and V_3 as follows:

V_1 : Vertex set containing 2 vertices, each of degree 2,

V_2 : Vertex set containing 2 vertices, each of degree 3 and

V_3 : Vertex set containing $(2n - 5)$ vertices, each of degree 4.

Thus $V(T(P_n); x) = 2x^2 + 2x^3 + (2n - 5)x^4$

Theorem 3.16: If $T(C_n)$ is the total graph of C_n , then

$$V(T(C_n); x) = 2nx^4$$

Proof: We know, a cycle graph C_n has n vertices. By definition of total graph of a graph, we note that, $T(C_n)$ of C_n is a 4-regular graph with $2n$ vertices and each vertex will be of degree 4. Thus $V(T(C_n); x) = 2nx^4$.

Corollary 3.17: The vertex polynomial root of $T(C_n)$ is 0 with multiplicity 4.

Proof: In view of Theorem 3.16, result is obvious.

Theorem 3.18: If $T(W_n)$ is the total graph of W_n , then

$$V(T(W_n); x) = (2n - 2)x^6 + (n - 1)x^{n+2} + x^{2n-2}, \text{ for } n \geq 5$$

Proof: We know a wheel graph W_n has n vertices and $2n - 2$ edges. By definition of total graph, we note that $T(W_n)$ of W_n will have $(3n - 2)$ vertices and we can partition this vertex set into three disjoint subsets V_1, V_2 and V_3 as follows:

V_1 : Vertex set containing $(2n - 2)$ vertices, each of degree 6,

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V_2 : Vertex set containing a single vertex of degree $(2n - 2)$
and

V_3 : Vertex set containing $(n - 1)$ vertices, each of degree $(n + 2)$.

Thus $V(T(W_n; x)) = (2n - 2)x^6 + (n - 1)x^{n+2} + x^{2n-2}$

IV. CONCLUSION

Vertex polynomial is one of the possible ways in which we can represent a graph algebraically. We can compare this polynomials with other polynomials associated with graph also.

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