Frictional Order Host-Vector Model for Transmission of Dengue Fever

Rashid Jan, Asif Jan

Abstract: ***Main purpose of this paper is to formulate an epidemiological model for dengue fever transmission using fractional order derivatives. Due to memory effect property, fractional order derivative has a benefit over the classical integer order models. This model for transmission of dengue fever of the non-integer order initial value problem will be based on the well-known fractional order Caputo derivative. Here our focus is on the existence of non-negative solutions of the frictional order dengue fever transmission model, furthermore, equilibria of the model and local asymptotic stability of model equilibria is investigated. In the end fractional order transmission model for dengue fever without immunity is presented.***

Keywords: ***Dengue fever, Caputo derivative, Existence of positive solution, Model equilibria, Asymptotic stability.***

I. INTRODUCTION

Mathematical models have been formulated, investigated and studied by many authors in engineering, finance, economics, science and in particular mathematical-biology(Infectious disease) using the classical integer order system of differential equations has received a lot of concentration in the past several years [1-14]. On the other hand due to the effective nature of fractional derivatives and integrals, many epidemiological models and other models in engineering and science have successful being originated and analyzed [15-26]. Fractional order derivatives has a significant characteristics called memory effect and this extraordinary property do not exist in the classical derivatives. These derivatives are nonlocal opposed to the local behavior of integer derivatives. It implies the next state of a fractional system depends not only upon its current state but also upon all of its historical states. Petras and Magin discussed in [27], "it is clear that the state of several systems(electrochemical, biological, etc.) at a given time depends on their configuration at previous times". The most important purpose of this paper is to originate and present a dengue fever transmission model with and without immunity using fractional order derivatives which has an advantage over the classical integer order models. Stability analysis of the model will also be a part of this paper.

II. MODEL FORMULATION

This part of paper deals with formulation of dengue disease transmission model of a non-integer order IVP (Initial Value Problem) using fractional order derivatives.

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Fractional calculus is an attractive and dominant tool for mathematical modeling. It has been applied in many areas of research such as economics, science and engineering. There are many interesting and attractive definitions of fractional derivatives in fractional calculus [26, 28], but here the famous Caputo derivatives is used due its advantage on initial value problems. Some important deifications related to frictional calculus are given below:

Definition 2.1. *[26, 28], Frictional integral of order α is defined by*

$$I^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(x)}{(t-x)^{1-\alpha}} dx$$

for 0< t, 0 < α < 1.

Definition 2.2. *[26, 28], Caputo Fractional derivative is defined by*

$$D^{\alpha} g(t) = \frac{1}{\Gamma(p-\alpha)} \int_0^t \frac{g^{p}(x)}{(t-x)^{\alpha+1-p}} dx,$$

for p – 1 <α < p.

A Host-Vector dengue disease transmission model was developed by Esteva and Vargas in [29], and suppose that a recover individual from the disease will not be reinfected by the disease. They also assume that the host population H is constant with death and birth rate μ_h. Where S_h, I_h, R_h are susceptible, infective, and recover individuals in the host population and S_v, I_v are susceptible, Infective in the vector population V. Their model is given as follows:

$$\frac{dS_h}{dt} = \mu_h H - \frac{\beta_h b}{H} S_h I_{\upsilon} - \mu_h S_h$$

$$\frac{dI_h}{dt} = \frac{\beta_h b}{H} S_h I_{\nu} - (\mu_h + \gamma_h) I_h$$

$$\frac{dR_h}{dt} = \gamma_h I_h - \mu_h R_h \qquad \textbf{(1)}$$

$$\frac{dS_{\nu}}{dt} = \wedge - \frac{\beta_{\nu} b}{H} S_{\nu} I_h - \mu_{\nu} S_{\nu}$$

$$\frac{dI_{\nu}}{dt} = \frac{\beta_{\nu} b}{H} S_{\nu} I_h - \mu_{\nu} I_{\nu},$$

where $\beta_h, \beta_{\upsilon}$ are the transmission probability from vector to host and host to vector. γ_h represent the recovery rate in the host population and b is the

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biting rate of the vector. Furthermore equation (1) can be reduced to three dimension dynamics with the condition $S_h + I_h + R_h = H$ and $S\upsilon + I\upsilon = \frac{\wedge}{\mu_\nu}$.

$$\frac{dS_h}{dt} = \mu_h H - \frac{\beta_h b}{H} S_h I_\upsilon - \mu_h S_h$$

$$\frac{dI_h}{dt} = \frac{\beta_h b}{H} S_h I_\nu - (\mu_h + \gamma_h) I_h \qquad \textbf{(2)}$$

$$\frac{dI_\nu}{dt} = \frac{\beta_\nu b}{H} S_\nu I_h - \mu_\nu I_{\nu,}$$

To normalize (2), we set $S = \frac{S_h}{H}, i = \frac{I_h}{H}, \upsilon = \frac{I_\nu}{\wedge / \mu_\nu}$

and get

$$\frac{d}{dt} s = \mu(1-s) - \rho s v$$
$$\frac{d}{dt} i = \rho s v - \beta i \qquad \textbf{(3)}$$
$$\frac{d}{dt} \upsilon = \gamma(1-\upsilon) i - \delta \upsilon,$$

Where

$$\rho = \frac{b\beta_h \wedge}{\mu_\upsilon H}, \beta = \gamma_h + \mu_h, \gamma = b\beta_{\upsilon,} \delta = \mu_{\upsilon,} \mu = \mu_h.$$

Replacing the integer derivatives in system (3) by Caputo derivatives of order α we obtain:

$$D^\alpha s = \mu(1-s) - \rho s \upsilon$$
$$D^\alpha i = \rho s \upsilon - \beta i \qquad \textbf{(4)}$$
$$D^\alpha z = \gamma(1-\upsilon) i - \delta \upsilon.$$

If we follow the method used by Diethelm [32], system (4) becomes

$$D^\alpha s = \mu^\alpha (1-s) - \rho^\alpha s \upsilon$$
$$D^\alpha i = \rho^\alpha s \upsilon - \beta^\alpha i \qquad \textbf{(5)}$$
$$D^\alpha \upsilon = \gamma^\alpha (1-\upsilon) i - \delta^\alpha \upsilon.$$

Now, it is understandable that the dimension on both sides of the system(5) correspond to $(times)^\alpha$. It is noticeable that when the fractional order α approaches to 1, then the model problem (5), becomes the classical endemic model (3).

III. NON-NEGATIVE SOLUTION

Let us assume $R^3 = \{Z \in R^3 : Z \geq 0\}$, where $Z = (s, i, \upsilon)^T$. To show the non-negative solution of the model we will apply the following lemma presented in [30].

Lemma: 3.1. *[30], Generalized Mean Value Theorem: Let* $g(x) \in C[c,d]$ *and* $D^a g(x) \in C[c,d]$ *for* $0 \leq \alpha \leq 1$, *then we have*

$$g(x) - g(c) + \frac{1}{\Gamma(\alpha)} D^\alpha g(\xi)(x-c)^\alpha,$$

with the condition $c \leq \xi \leq x$, *for all* $x \in [c,d]$.

Remark 3.1: Assume that $g(x) \in C[c, d]$ and $D^\alpha g(x) \in C[c, d]$, for $0 \leq \alpha \leq 1$. It follows form lemma 3.1 that $g(x)$ is non-decreasing if $D^\alpha g(x) \geq 0$, for all $x \in [c, d]$ and $g(x)$ non increasing if $D^\alpha g(x) \leq 0$ for all $x \in [c,d]$.

Theorem 3.1 *A unique solution of the fractional order initial value problem (5) exists and it remains in* $R^3_\dagger$. *on each hyperplane bounding the non negative orthant, the vector field points into* R^3.

Proof 3.1 *Existence and uniqueness of the solution of model problem (5) in (0, ∞) follows by the use of theorem 3.1 and remark 3.2 in [31]. The domain* R^3 *is positively invariant for the model problem, because*

$$D^\alpha s\big/_{s=0} = \mu \geq 0$$
$$D^\alpha i\big/_{i=0} = 0$$
$$D^\alpha \upsilon\big/_{\upsilon=0} = \gamma \geq 0,$$

on each hyper plane bounding the non negative outhunt, the vector field points into R^3+.

IV. EQUILIBRIA OF THE MODEL

The frictional order model (5) has two biological meaningful equilibrium points i.e. disease free and endemic equilibrium. These two equilibria depending on i and υ: if there is no disease for host and mosquitoes i.e. if i = υ = 0, then the equilibrium point is said to be disease free equilibrium, and if $i \neq 0$, $\upsilon \neq 0$, then the equilibrium point is called endemic. To determine the equilibria of the frictional order model (5). Assume that

$D^\alpha s = 0, D^\alpha i = 0, D^\alpha \upsilon = 0$, we get

$$0 = \mu^\alpha (1-s) - \rho^\alpha s \upsilon$$
$$0 = \rho^\alpha s \upsilon - \beta^\alpha i$$
$$0 = \gamma^\alpha (1-\upsilon) i - \delta^\alpha \upsilon,$$

then the disease free equilibrium $E_f(1, 0, 0)$ and endemic equilibrium $E_e(s^*, i^*, \upsilon^*)$, where

$$s^* = \frac{\mu^\alpha \gamma^\alpha + \beta^\alpha \delta^\alpha}{\gamma^\alpha (\mu^\alpha + \rho^\alpha)}, i^* = \frac{\mu^\alpha (\rho^\alpha \gamma^\alpha - \beta^\alpha \delta^\alpha)}{\beta^\alpha \gamma^\alpha (\mu^\alpha + \rho^\alpha)},$$

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and $\upsilon^* = \frac{\mu^\alpha(\rho^\alpha\gamma^\alpha - \beta^\alpha\delta^\alpha)}{\rho^\alpha(\mu^\alpha\gamma^\alpha + \beta^\alpha\delta^\alpha}$. by and

The second fixed point exists only if the threshold parameter

$$R = \frac{\rho^\alpha\gamma^\alpha}{\beta^\alpha\delta^\alpha} > 0,$$

the basic reproductive number is denoted by R_0 and is equal to $\sqrt{R}$.

Theorem 4.1 *The disease free equilibrium E_f is globally stable if $R < 1$. If $R > 1$, then the endemic fixed point E_e is globally asymptotically stable and E_f is unstable.*

It is proved in [29] that disease free equilibrium point is globally stable if $R < 1$.For $R > 1$ the fixed point E_e becomes locally asymptotically stable and the fixed point E_f becomes unstable. The global stability is shown using the property of stability of periodic orbits.

V. MODEL WITHOUT IMMUNITY

There is no enough information about the immunity after recovery for Dengue Fever, that's why we suppose that the immune individuals in the host population is negligible. In this case the Host-Vector Model (1) for Dengue Transmission becomes as:

$$\begin{aligned}
\frac{ds_h}{dt} &= \mu_h H - \frac{\beta_h b}{H} S_h I_\upsilon - \mu_h S_h \\
\frac{dI_h}{dt} &= \frac{\beta_h b}{H} S_h I_\upsilon - (\mu_h + \gamma_h) I_h \\
\frac{dS_\upsilon}{dt} &= \wedge - \frac{\beta_\upsilon b}{H} S_\upsilon I_h - \mu_\upsilon S_\upsilon \\
\frac{dI_\upsilon}{dt} &= \frac{\beta_\upsilon b}{H} S_\upsilon I_h - \mu_u I_\upsilon,
\end{aligned} \qquad (6)$$

where $S_h + I_h = H$ and $S_\upsilon + I_\upsilon = \frac{\wedge}{\mu_\upsilon}$ Simplifying in the same way, we get

$$\begin{aligned}
\frac{d}{dt} i &= \rho(1-i)\upsilon - \beta i \\
\frac{d}{dt} \upsilon &= \gamma(1-\upsilon)i - \delta\upsilon.
\end{aligned} \qquad (7)$$

Replacing the integer derivatives in system (7) by Caputo derivatives of order a then the system can be written as:

$$\begin{aligned}
D^\alpha i &= \rho(1-i)\upsilon - \beta i \\
D^\alpha \upsilon &= \gamma(1-\upsilon)i - \delta\upsilon,
\end{aligned} \qquad (8)$$

Using the same technique used by Diethelm [32], system (8) becomes

$$\begin{aligned}
D^\alpha i &= \rho^\alpha(1-i)\upsilon - \beta^\alpha i \\
D^\alpha z &= \gamma^\alpha(1-\upsilon)i - \delta^\alpha\upsilon
\end{aligned} \qquad (9)$$

In this case we have two fixed point (0,0) and (i^*, υ^*), Where $i^* = \frac{\rho^\alpha\gamma^\alpha - \beta^\alpha\delta^\alpha}{\gamma^\alpha(\rho^\alpha + \beta^\alpha)}$ and $\upsilon^* = \frac{\rho^\alpha\gamma^\alpha - \beta^\alpha\delta^\alpha}{\rho^\alpha(\gamma^\alpha + \delta^\alpha}$

It is simple and easy to and the basic reproduction number for system (9). The origin is locally stable if the basic reproduction number is less than one and unstable if the basic reproduction number greater than one.

VI. CONCLUSION

In this research we formulated a frictional order Host-vector model for dengue fever. In our work we established the existence and uniqueness of non-negative solutions of the frictional order model. We have shown that there are two biological meaningful equilibria of the system and proved that the disease free equilibrium E_f is globally stable if the basic reproduction number less than one. If the basic reproduction number greater than one, then the endemic fixed point E_e is globally asymptotically stable and E_f is unstable. In last section the model is analyzed and formulated without immunity.

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